ON THE OPTIMAL STABILIZATION OF A RIGID BODY WITH A FIXED POINT BY MEANS OF PENDULUMS

(OB OPTIMALNOI STABILIZATSII TVERDEGO TELA S NEPODVIZHNOI TOCHKOI PRI POMOSHCHI MAKHOVIKOV)

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V.V. KREMENTULO
(Moscow)

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We present here a solution of the problem of how to construct analytically controls which give the optimal stability to a rigid body with a fixed point. We apply a theorem related to the direct method of Liapunov.

We consider a mechanical system consisting of a rigid body (a platform) with a fixed point. Its principal axes of inertia coincide with the axes of three homogeneous symmetric pendulums. These pendulums are set in motion by special motors. Such a system can be regarded as a gyrostat because its distribution of mass does not change in the process of motion. Let the fixed point O coincide with the center of mass; let $OX_1X_2X_3$ be the fixed coordinate system; $Ox_1x_2x_3$ be the moving coordinate system attached to the body and coinciding with the principal axes of inertia (axes of the pendulums).

TABLE 1

	x_1	x_2	x_3
X_1	α_{11}	α_{12}	α_{13}
X_2	α_{21}	α_{22}	α_{23}
X_3	α_{31}	α_{32}	α_{33}

Let us introduce the following notations: p_1, p_2, p_3 are the projections of the absolute instantaneous angular velocity, on the x_1, x_2, and x_3 axis respectively, C_1, C_2, and C_3, are the moments of inertia of the system about the x_1, x_2 and x_3 axis respectively, J_1, J_2, and J_3, are the axial moments of inertia of the pendulums, and ω_1, ω_2, and ω_3, are their relative angular velocities. The direction cosines between the axes $OX_1X_2X_3$ and $Ox_1x_2x_3$ are shown in the table on the left.

1. The statement of the problem. The initial equations of motion. The equations of motion of our system are written in the form of the three Eulerian dynamical equations

$$C_1 \frac{dp_1}{dt} + (C_3 - C_2)\, p_2 p_3 + p_2 H_3 - p_3 H_2 + \frac{dH_1}{dt} = 0 \qquad (123)$$
$$(H_i = J_i \omega_i, \quad i = 1, 2, 3) \qquad (1.1)$$

The symbol (123) indicates that the remaining equations are obtained by cyclic permutation. The equation (1.1) are followed by the nine Poisson kinematic equations

$$\frac{d\alpha_{i1}}{dt} + p_2\alpha_{i3} - p_3\alpha_{i2} = 0 \quad (i = 1,\ 2,\ 3) \quad (123) \tag{1.2}$$

We must take into account that the nine variables α_{ik} $(i,\ k = 1,\ 2,\ 3)$ are connected by six geometric relations

$$\sum_i \alpha_{ki}\alpha_{li} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases} \quad (k,\ l = 1,\ 2,\ 3) \tag{1.3}$$

Both here and further the summation is performed from 1 to 3. In addition to (1.1) and (1.2) we shall introduce three additional equations describing the rotational motion of the pendulums. Neglecting internal friction these equations have the form

$$J_i(\omega_i^{\cdot} + p_i^{\cdot}) = u_i \qquad (i = 1,\ 2,\ 3) \tag{1.4}$$

where u_1, u_2, and u_3 are the controlling moments generated by the motors.

The systems of equations (1.1) to (1.4) describe completely the motion of our mechanical system. The obtained equations of motion permit a particular solution corresponding to the position of equilibrium of the principal body (platform) with the controls switched off $(u_i = 0)$:

$$p_i = 0, \qquad \alpha_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad (i,\ k = 1,\ 2,\ 3), \quad \omega_i = \omega_i^{\circ} \tag{1.5}$$

We shall try to solve the problem of partial optimal stabilization of the position of equilibrium (1.5); to do this we shall have to choose u_i as functions of p_1, p_2, p_3, $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{33}$, so as to ensure that, if initial perturbations are sufficiently small, then the platform will asymptotically move back to the initial position

$$p_i = 0, \quad \alpha_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad (i, k = 1, 2, 3) \tag{1.6}$$

This function should also minimise a functional which fully describes the character of the transitional process. Also, the angular velocities of the pendulums ω_i may not reach their initial values ω_i°. We shall assume $\omega_i^{\circ} = 0$. Our problem can be regarded for example as the problem of stabilization of the position of equilibrium of an artificial satellite rotating about the center of mass.

The angular velocities ω_i can, in general, be eliminared from our considerations, in analogy with eliminating the cyclic velocities in analytical mechanics. In our problem the vector of the angular momentum of the system about the point O is invariant, that is $\mathrm{G} = \mathrm{const}$, and projected on the $OX_1X_2X_3$ axes it gives

$$\sum_i (C_i p_i + H_i)\,\alpha_{ki} = h_k = \mathrm{const} \qquad (k = 1, 2, 3) \tag{1.7}$$

Since the determinant of the system (1.7) is equal to 1, we can solve the system with respect to $C_i p_i + H_i$ $(i = 1, 2, 3)$. We obtain

$$C_i p_i + J_i \omega_i = Q_i(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{33}, h_1, h_2, h_3) \qquad (i = 1,2,3) \tag{1.8}$$

where Q_i denotes the determinant of the system (1.7) in which the i-th column is replaced by the column from h_k. Differentiating (1.8) with respect to time and using the obtained expressions we shall eliminate $\omega_i^{\cdot}$ from the equations (1.4). At the same time we shall replace the derivatives $\alpha_{ik}^{\cdot}$ $(i, k = 1, 2, 3)$, by expressions from (1.2). This will result in the following system of three equations

$$(C_1 - J_1)\frac{dp_1}{dt} = -p_2(h_1\alpha_{13} + h_2\alpha_{23} + h_3\alpha_{33}) + \\ + p_3(h_1\alpha_{12} + h_2\alpha_{22} + h_3\alpha_{32}) - u_1 \ (123) \tag{1.9}$$

taking into account that

$$\alpha_{21}\alpha_{32} - \alpha_{31}\alpha_{22} = \alpha_{13} \quad (123) \tag{1.91}$$

In this way our mechanical system is described by twelve equations (1.2) and (1.9), with the variables connected by six equations (1.3). The angular velocities ω_i do not appear in the obtained equations, consequently we have to solve the conventional problem of optimal stabilization of the position of equilibrium (1.6). The phase coordinates are $p_1, p_2, p_3, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{33}$, and by (1.3) only six of them are independent.

2. Solution of the problem of optimal stabilization of a platform. Assuming (1.6) to be the equations of the unperturbed motion we shall construct the equations of the perturbed motion, preserving the symbolism throughout for the initial variables and the perturbations.

$$\frac{dp_1}{dt} = -h_{13}p_2 + h_{12}p_3 + P_1(p_2, p_3, \alpha_{11}, \ldots, \alpha_{33}) + v_1 \qquad (123) \tag{2.1}$$

$$\frac{d\alpha_{ii}}{dt} = A_{ii} \quad (i = 1, 2, 3); \qquad \frac{d\alpha_{12}}{dt} = -p_3 + A_{12}, \frac{d\alpha_{13}}{dt} = p_2 + A_{13} \quad (123)$$

$$h_{ik} = \frac{h_k}{C_i - J_i}, \quad v_i = -\frac{u_i}{C_i - J_i}, \quad A_{i1} = p_3\alpha_{i2} - p_2\alpha_{i3} \quad (123) \quad (i, k = 1, 2, 3) \tag{2.2}$$

Here P_1, P_2, P_3 are the omitted terms of the second order of smallness. The quantities h_{ik} characterizing the initial perturbations will be regarded as small constant parameters.

In addition to the equations (2.1) we shall also introduce the system of equations of the first approximations

$$\frac{dp_1}{dt} = -h_{13}p_2 + h_{12}p_3 + v_1 \qquad (123) \tag{2.3}$$

The system (2.3) and (2.2) will be called the 'reduced' system as compared with the full system (2.1), (2.2).

The problem consists of the following: to determine functions v_i of the phase coordinates so as to make the null solution

$$p_i = 0, \quad \alpha_{ik} = 0 \qquad (i, k = 1, 2, 3) \tag{2.4}$$

asymptotically stable and to fulfill the condition for a minimum of the integral

$$\int_{t_0}^{\infty} \Omega(p_1, p_2, p_3, \alpha_{11}, \ldots, \alpha_{33}, v_1, v_2, v_3)\, dt \tag{2.41}$$

where Ω is a positive function which shall be determined in the process of solving the problem. At present we shall only define the form of Ω, by putting (2.5)

$$\Omega = F_1(p_1, p_2, p_3) + F_2(\alpha_{11}, \ldots, \alpha_{33}) + \sum n_i v_i^2 + \Lambda(p_1, p_2, p_3, \alpha_{11}, \ldots, \alpha_{33})$$

where

$$F_1(p_1, p_2, p_3) = \sum_{i,k} e_{ik} p_i p_k \tag{2.51}$$

and remembering that later on we shall impose certain suitable restrictions on the constant coefficients e_{ik} $(e_{ii} > 0)$, and $n_i > 0$ $(i, k = 1, 2, 3)$; also, the function F_2 has to be defined, Λ denotes the possible terms of the order higher than two. The function F_1 should be a positive-definite quadratic form of velocities, which the function F_2 will be assumed to be a positive (positive-definite if possible) quadratic form of α_{ik}.

To solve our problem we shall use the fundamental theorem of the second Liapunov's method of investigation of the problems of optimal stabilization (see for example [1]). This theorem gives the sufficient conditions for optimal stabilization and is based on the Liapunov theorem on the asymptotic stability and on the partial differential equations due to Bellman.

At the beginning we shall consider the problem of optimal stabilization of the unperturbed motion (2.4) on the strength of the 'reduced' system of equations. By the theorem, the optimal control v_i° and the optimal Liapunov functions V° should satisfy the following system of four equations

$$\begin{gathered}\frac{\partial V^\circ}{\partial p_1}(-h_{13}p_2 + h_{12}p_3 + v_1^\circ) + \frac{\partial V^\circ}{\partial p_2}(h_{23}p_1 - h_{21}p_3 + v_2^\circ) + \\ + \frac{\partial V^\circ}{\partial p_3}(-h_{32}p_1 + h_{31}p_2 + v_3^\circ) + \sum_i \delta_i p_i + \sum_{i,k} \frac{\partial V^\circ}{\partial \alpha_{ik}} A_{ik} + \\ + \Omega(p_1, p_2, p_3, \alpha_{11}, \ldots, \alpha_{33}, v_1^\circ, v_2^\circ, v_3^\circ) = 0 \\ \frac{\partial V^\circ}{\partial p_i} + 2n_i v_i^\circ = 0 \qquad (i = 1, 2, 3)\end{gathered} \tag{2.6}$$

Where

$$\delta_1 = \frac{\partial V^\circ}{\partial \alpha_{32}} - \frac{\partial V^\circ}{\partial \alpha_{23}} \quad (123) \tag{2.61}$$

Since

$$v_i^\circ = -\frac{1}{2n_i}\frac{\partial V^\circ}{\partial p_i} \qquad (i = 1, 2, 3) \tag{2.7}$$

we obtain one first order non-linear partial differential equation

$$-\sum_i \frac{1}{4n_i}\left(\frac{\partial V^\circ}{\partial p_i}\right)^2 + \frac{\partial V^\circ}{\partial p_1}(-h_{13}p_2 + h_{12}p_3) + \frac{\partial V^\circ}{\partial p_2}(h_{23}p_1 - h_{21}p_3) +$$
$$+ \frac{\partial V^\circ}{\partial p_3}(-h_{32}p_1 + h_{31}p_2) + \sum_i \delta_i p_i + \sum_{i,k}\frac{\partial V^\circ}{\partial \alpha_{ik}}A_{ik} + F_1(p_1, p_2, p_3) + \tag{2.8}$$
$$+ F_2(\alpha_{11}, \ldots, \alpha_{33}) + \Lambda(p_1, p_2, p_3, \alpha_{11}, \ldots, \alpha_{33}) = 0$$

determining V°.

By (1.3) the perturbations α_{ik} $(i, k = 1, 2, 3)$ are connected by six equations

$$\Phi_{kl} = \alpha_{kl} + \alpha_{lk} + \sum_i \alpha_{ki}\alpha_{li} = 0 \qquad (k, l = 1, 2, 3;\ k \leqslant l) \tag{2.9}$$

which can be regarded as integrals of the equations (2.2) and (2.3).

We shall now introduce a function Φ_0 with undetermined coefficients consisting of quadratic and linear terms

$$2\Phi_0 = -2\sum_i k_i\alpha_{ii} + \sum_i m_i p_i^2 + 2p_1\sum_{i,k} a_{ik}\alpha_{ik} + 2p_2\sum_{i,k} b_{ik}\alpha_{ik} + 2p_3\sum_{i,k} c_{ik}\alpha_{ik} \tag{2.10}$$
$$(k_i > 0,\ m_i > 0)$$

we assume that the function V° is of the form

$$2V^\circ = 2\Phi_0 + \sum_i k_i\Phi_{ii} \tag{2.11}$$

that is, it represents a quadratic form in all the variables $p_1, p_2, p_3, \alpha_{11}, \ldots, \alpha_{33}$. Partial derivatives $\partial V^\circ / \partial p_i$, which by (2.7), express the control, are

$$\frac{\partial V^\circ}{\partial p_1} = m_1 p_1 + \sum_{i,k} a_{ik}\alpha_{ik}, \quad \frac{\partial V^\circ}{\partial p_2} = m_2 p_2 + \sum_{i,k} b_{ik}\alpha_{ik}, \quad \frac{\partial V^\circ}{\partial p_3} = m_3 p_3 + \sum_{i,k} c_{ik}\alpha_{ik} \tag{2.12}$$

Substituting (2.1) into (2.8) and equating the coefficients of the second order terms to zero, we obtain a system of algebraic equations connecting the coefficients of the functions V° and Ω

$$d_1^2 n_1 + a_{23} - a_{32} = e_{11}, \quad d_2^2 n_2 - b_{13} + b_{31} = e_{22}, \quad d_3^2 n_3 + c_{12} - c_{21} = e_{33}$$
$$m_1 h_{13} - m_2 h_{23} = 2e_{12}, \quad -m_1 h_{12} + m_3 h_{32} = 2e_{13}, \quad m_2 h_{21} - m_3 h_{31} = 2e_{23} \tag{2.13}$$
$$d_i = m_i / 2n_i \qquad (i = 1,2,3)$$

The remaining equations break down into nine sets, linear with respect to a_{ik}, b_{ik}, and c_{ik}. Each of these sets contains three coefficients corresponding to the subscripts $i = k$.

All sets have the same determinant

$$\Delta_1 = \begin{vmatrix} -d_1, & h_{23}, & -h_{32} \\ -h_{13}, & -d_2, & h_{31} \\ h_{12}, & -h_{21}, & -d_3 \end{vmatrix} \tag{2.14}$$

while their right hand side terms contain numbers k_i $(i = 1, 2, 3)$.

We shall assume the numbers d_i to be given and sufficiently large (the lower bound for d_i will be determined later). Then, since h_{ik} is small the determinant Δ_1 will not be equal to zero and every set will have a unique solution. These are:

$$a_{11} = b_{11} = c_{11} = 0$$

$$a_{12} = -\frac{k_1}{\Delta_1}(d_2h_{32} - h_{23}h_{31}), \qquad a_{13} = -\frac{k_1}{\Delta_1}(d_3h_{23} + h_{21}h_{32})$$

$$b_{12} = \frac{k_1}{\Delta_1}(d_1h_{31} + h_{13}h_{32}), \qquad b_{13} = -\frac{k_1}{\Delta_1}(d_1d_3 + h_{12}h_{32})$$

$$c_{12} = \frac{k_1}{\Delta_1}(d_1d_2 + h_{13}h_{23}), \qquad c_{13} = \frac{k_1}{\Delta_1}(d_1h_{21} - h_{12}h_{23})$$

$$a_{22} = b_{22} = c_{22} = 0$$

$$a_{21} = \frac{k_2}{\Delta_1}(d_2h_{32} - h_{23}h_{31}), \qquad a_{23} = \frac{k_2}{\Delta_1}(d_2d_3 + h_{21}h_{31})$$

$$b_{21} = -\frac{k_2}{\Delta_1}(d_1h_{31} + h_{13}h_{32}), \qquad b_{23} = -\frac{k_2}{\Delta_1}(d_3h_{13} - h_{12}h_{31}) \tag{2.15}$$

$$c_{21} = -\frac{k_2}{\Delta_1}(d_1d_2 + h_{13}h_{23}), \qquad c_{23} = \frac{k_2}{\Delta_1}(d_2h_{12} + h_{13}h_{21})$$

$$a_{33} = b_{33} = c_{33} = 0$$

$$a_{31} = \frac{k_3}{\Delta_1}(d_3h_{23} + h_{21}h_{32}), \qquad a_{32} = -\frac{k_3}{\Delta_1}(d_2d_3 + h_{21}h_{31})$$

$$b_{31} = \frac{k_3}{\Delta_1}(d_1d_3 + h_{12}h_{32}), \qquad b_{32} = \frac{k_3}{\Delta_1}(d_2h_{13} - h_{12}h_{31})$$

$$c_{31} = -\frac{k_3}{\Delta_1}(d_1h_{21} - h_{12}h_{23}), \qquad c_{32} = -\frac{k_3}{\Delta_1}(d_2h_{12} + h_{13}h_{21})$$

The formulas (2.15) show that, when d_i are sufficiently large, then all a_{ik}, b_{ik}, and c_{ik} $(i, k = 1, 2, 3)$ being of the order $1/d_i$ are sufficiently small. This will secure a positive-definiteness of the function V°. At the same time the function $F_1(p_1, p_2, p_3)$, will also be positive-definite, since by (2.13) the coefficients e_{ik} $(i \neq k)$ are small in comparison with e_{ii}. The function F_2 is given in the form

$$F_2(\alpha_{11}, \ldots, \alpha_{33}) = \frac{1}{4n_1}\Big(\sum_{i,k} a_{ik}\alpha_{ik}\Big)^2 + \frac{1}{4n_2}\Big(\sum_{i,k} b_{ik}\alpha_{ik}\Big)^2 + \frac{1}{4n_3}\Big(\sum_{i,k} c_{ik}\alpha_{ik}\Big)^2 \tag{2.16}$$

Since α_{ik} $(i, k = 1, 2, 3)$ are connected by six equations (2.9) the function F_2 can be made positive-definite. The region $F_2 = 0$ is determined by the equations

$$\Phi_{21} = \sum_{i,k} a_{ik}\alpha_{ik} = 0, \quad \Phi_{31} = \sum_{i,k} b_{ik}\alpha_{ik} = 0, \quad \Phi_{32} = \sum_{i,k} c_{ik}\alpha_{ik} = 0 \quad (2.17)$$

Together with (2.9) we obtain nine relations

$$\Phi_{kl} = 0 \qquad (k, l = 1, 2, 3) \qquad (2.171)$$

The function F_2 will be positive-definite if the position of equilibrium (2.4) is isolated. The latter takes place when

$$\left[\frac{D(\Phi_{11}, \Phi_{12}, \ldots, \Phi_{33})}{D(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{33})}\right]_{\alpha_{11}=\ldots=\alpha_{33}=0} \neq 0 \qquad (2.18)$$

By (2.15) the condition (2.18) can be written in the form

$$\Delta_2 = \begin{vmatrix} d_2d_3 + h_{21}h_{31} & -d_3h_{23} - h_{31}h_{32} & -d_2h_{32} + h_{23}h_{31} \\ d_3h_{13} - h_{12}h_{31} & d_1d_3 + h_{12}h_{32} & -d_1h_{31} - h_{13}h_{32} \\ d_2h_{12} + h_{13}h_{21} & d_1h_{21} - h_{12}h_{23} & d_1d_2 + h_{13}h_{23} \end{vmatrix} \neq 0 \qquad (2.19)$$

The above relation is certainly satisfied when d_i is sufficiently large.

In order for the equation (2.8) to be exactly satisfied for a given function V°, we must write the function Λ from (2.5) in the form

$$\Lambda(p_1, p_2, p_3, \alpha_{11}, \ldots, \alpha_{33}) = -\sum_{i,k}(p_1a_{ik} + p_2b_{ik} + p_3c_{ik}) A_{ik} \qquad (2.20)$$

Addition of this function does obviously not alter the sign-definiteness of the basic quadratic form.

Note that at large values of d_i the relations (2.13) give

$$d_i \approx \sqrt{e_{ii} / n_i} \qquad (i = 1, 2, 3) \qquad (2.201)$$

which means that large values of d_i are equivalent to large values of e_{ii}/n_i.

In this manner we have shown that the position of equilibrium (2.4) is stabilized on the strength of the 'reduced' system of equations (2.3), (2.2) by the controls (2.7), (2.12), and (2.15), if the constants k_i, m_i, and n_i, are such that: (1) the forms V° and F_1 are positive-definite; (2) the conditions $\Delta_1 \neq 0$, and $\Delta_2 \neq 0$ are satisfied.

From these conditions and with fixed k_i we can calculate the lower bound for $m_i / 2n_i$. Besides, the control obtained turns out to be optimal with regard to the minimum of the integral of the functions Ω, (2.5), (2.13) and (2.20).

It is easily seen that from the stabilization of the position of equilibrium (2.4) on the strength of the 'reduced' system of equations (2.3) and (2.2) implies the stabilization of (2.4) on the strength of the full system of equations (2.1) and (2.2). Indeed, in near vicinity of (2.4), the terms P_1, P_2, and P_3 in (2.1) satisfy the conditions

$$|P_1| < \varepsilon_1 l_1 (|p_2| + |p_3|), \quad l_1 = \max(|h_{12}|, |h_{13}|) \quad (123) \tag{2.21}$$

Here ε_1, ε_2, and ε_3 are sufficiently small positive constants.

The function V° which we constructed,determines the symptotic stability of the solution (2.4) of the full system of equations (2.1) and (2.2), since the time derivative of V° from the full equations differs from the corresponding derivative from the 'reduced' equations only by additional terms of higher order

$$\sum_i \frac{\partial V^\circ}{\partial p_i} P_i \tag{2.22}$$

which, by (2.21), does not change its sign-definiteness, arising from the 'reduced' system.

Consequently, the derived equation secures the stabilization of (2.4), taking into account the full system of equations (2.1), and (2.2). This stabilization, however, would not be optimal in the sense of minimizing the integral of the functions (2.5), (2.13) and (2.20), because the equation (2.8) would not be satisfied on account of the appearance of the new terms (2.22). Nevertheless, it is possible to secure the optimal stabilization (2.4), by adding to the derived functions V° and Ω suitable terms of high order, which would take care of the terms (2.22). This method is not unique. For example we can, without changing V°, add all the terms of (2.22) with the opposite sign.

3. **Analysis of the obtained results.** According to (2.7), (2.12) and (2.15) the obtained control v_i° (the control u_i° differs from v_i° by the factors $J_i - C_i$)

$$-v_1^\circ = d_1 p_1 + \frac{1}{2n_1}\sum_{i,k} a_{ik}\alpha_{ik}, \qquad -v_2^\circ = d_2 p_2 + \frac{1}{2n_2}\sum_{i,k} b_{ik}\alpha_{ik}$$
$$-v_3^\circ = d_3 p_3 + \frac{1}{2n_3}\sum_{i,k} c_{ik}\alpha_{ik} \tag{3.1}$$

has the property that the terms which are linear with respect to the velocities p_1, p_2, and p_3 and which have large coefficients d_i can be separated from the terms depending on the coordinates α_{ik} whose coefficients are expressed in terms of d_i and of the initial perturbations h_{ik}. The equation (3.1) depends essentially on the initial perturbations ; the greater h_{ik}, the larger values of d_i must be chosen. We shall assume that h_{ik} are suitably small, and that their magnitude is of the order of α_{ik}. Without loss of generality we can assume $k_1 = k_2 = k_3 = k > 0$. (It was never assumed that k_i were distinct). Then, by (2.15) and taking into account $\Delta_1 \approx - d_1 d_2 d_3$, the control (3.1) can be put into the following form

$$-v_1^\circ = d_1 p_1 + \frac{k}{m_1}(\alpha_{32} - \alpha_{23}) + [2]_1 + [3]_1 \quad (123) \tag{3.2}$$

where the symbols $[2]_i$ and $[3]_i$ denote the omitted terms of the second and third order of smallness (taking into account the smallness of h_{ik}). Thus, the terms in (3.1) which depend on the coordinates, can be separated into terms of the first, second and third order of smallness.

It is useful to write d own the control (3.2) in form of the functions of the angular deviations of the platform. For example let the coordinates determining the orientation of the platform be the three Krylov angles [2]: φ (yawing), ψ (trim angle), θ (rolling). The investigated position of equilibrium (2.4) has the form

$$\varphi = \psi = \theta = 0 \tag{3.3}$$

The perturbation α_{ik} are expressed in terms of the perturbations of the angles φ, ψ, θ according to the formulas

$$\begin{aligned} \alpha_{12} &= -\varphi + \ldots, \quad \alpha_{13} = \psi + \ldots, \quad \alpha_{23} = -\theta + \ldots \\ \alpha_{21} &= \varphi + \ldots, \quad \alpha_{31} = -\psi + \ldots, \quad \alpha_{32} = \theta + \ldots \end{aligned} \tag{3.4}$$

The dots denote the terms of the order of smallness higher than one. From (3.2) and (3.4) we obtain

$$\begin{gathered} -v_1^\circ = d_1 p_1 + \frac{2k}{m_1}\theta + \ldots, \qquad -v_2^\circ = d_2 p_2 + \frac{2k}{m_2}\psi + \ldots \\ -v_3^\circ = d_3 p_3 + \frac{2k}{m_3}\varphi + \ldots \end{gathered} \tag{3.5}$$

The structure of the control equation (3.1) can be assumed in advance, by taking the following considerations. Let us consider the control

$$v_i^* = -d_i p_i \qquad (d_i > 0, \ i = 1, 2, 3) \tag{3.6}$$

It can be shown that we can select d_i in such a manner that all the roots of the characteristic equation of the system (2.3) will have negative real parts. The remaining roots of the characteristic equation of the first approximation system corresponding to (2.3), and (2.2) will be equal to zero. Consequently we can formally obtain the critical case of the nine zero roots. A complete and working solution of the problem of stability in this case has not yet been obtained. However, the right hand side members of the system (2.2) are such, that when the non-critical variables p_1, p_2, and p_3 vanish, they also vanish. That means that the last critical case is a 'particular' one [3], for which the problem of stability is easily solved. The equation (2.4) is stable, and asymptocially stable with respect to the velocities p_1, p_2 and p_3. This means that the platform approaches asymptotically one of the positions of equilibrium, which is in the neighborhood of (2.4). It means also that for the position of equilibrium (2.4) the control (3.6) secures the asymptotic stability with respect to the velocities and the ordinary stability with respect to the coordinates. To make (2.4) asymptotically stable with respect to the coordinates as well, it is sufficient to add to the control (3.6) small terms which are coordinate dependent.

4. Effect of internal friction. In our investigations we neglected the viscous friction in the axes of the pendulums, which in real systems is always present and could influence essentially their motion. The equation (3.1) shows that the platform after initial perturbation returns to the initial position of equilibrium (1.6). In this position the controlling moments u_i vanish and the pendulums continue to rotate by inertia. However, the moments of the viscous friction in the axes $x_1 x_2 x_3$

$$M_i = -f_i \omega_i \qquad (f_i > 0, \ i = 1, 2, 3) \tag{4.1}$$

being not compensated, will cause the platform to move away from the achieved position of equilibrium. To prevent this, the controlling moments u_i in the position of equilibrium (1.6) should not vanish, but should balance the resisting moments (4.1). Instead of the equations (4.1) we shall then have

$$J_i (\omega_i^{\cdot} + p_i^{\cdot}) = u_i + M_i \qquad (i = 1, 2, 3) \tag{4.2}$$

In the position of equilibrium (1.6) the moments $u_{i0} = f_i \omega_i$. By taking

$$v_i = -\frac{1}{C_i - J_i} (u_i - f_i \omega_i) \qquad (i = 1, 2, 3) \tag{4.3}$$

we are able to obtain the following results. The required expression for u_i° becomes

$$u_i^\circ = f_i \omega_i - (C_i - J_i) v_i^\circ \qquad (i = 1, 2, 3) \tag{4.4}$$

where ω_i are functions of $p_1, p_2, p_3, \alpha_{11}, \ldots, \alpha_{33}, h_1, h_2$, and h_3 in agreement with (1.8), and v_i° are obtained in the form (3.2).

Finally we note that the obtained results remain valid also when the perturbations h_{ik} are finite. Selecting d_i sufficiently large we can satisfy all the inequalities given above and write the control equation in the form (3.2) or (4.4).

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